Hadamard Gate in Quantum Mechanics

- The Hadamard gate is a single-qubit quantum gate that creates superposition. It is represented by the matrix
- $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
- Action on Basis States:

$$H\,|0\rangle = (1\,/\,\sqrt{2})\,(|0\rangle + |1\rangle)$$
$$H\,|1\rangle = (1\,/\,\sqrt{2})\,(|0\rangle - |1\rangle)$$

Eigenvalues and Eigenvectors of H

- **Eigenvalues**: These tell us how the gate scales certain vectors.
  - We solve $\det(H - \lambda I) = 0$.
  - The result is $\lambda = \pm 1$.

- **Eigenvectors**: These are the specific vectors that are scaled by the eigenvalues.

For $\lambda = +1$, the normalized eigenvector is $\frac{1}{\sqrt{4-2\sqrt{2}}}\begin{bmatrix} 1 \\ \sqrt{2} - 1 \end{bmatrix}$.

For $\lambda = -1$, the normalized eigenvector is $\frac{1}{\sqrt{4-2\sqrt{2}}}\begin{bmatrix} 1 \\ -\sqrt{2} - 1 \end{bmatrix}$

Eigenvalues and Eigenvectors of H

- The Hadamard gate can be viewed as an operation driven by a **Hamiltonian** $\left(\widehat{H}\right)$
- **The Hamiltonian** represents the total energy of a quantum system.
- It governs a system's **time evolution** via the Schrödinger equation:
- $i\hbar\ (d/dt)\ |\psi(t)\rangle\ =\ \hat{H}\ |\psi(t)\rangle$
- The time evolution operator is $\bigcup(t) = e^{-i\widehat{H}/\hbar}$
- We can say that H= $e^{-i\widehat{H}/\hbar}$ for some Hamiltonian $\hat{H}$ and a specific time t.

Eigenvalues and Eigenvectors of H

- We can use the eigenvalues of H to find the eigenvalues of the underlying Hamiltonian.
- H eigenvalue = $e^{-i\hat{H}/\hbar}$
- **Deriving for $\boldsymbol{\lambda = +1}$:**

  $+1 = e^{-i\lambda_1/\hbar}$ only if $\lambda_1 = 0$

- **Deriving for $\boldsymbol{\lambda = -1}$:**

  $-1 = e^{-i\lambda_1/\hbar}$ only if $\lambda_1 = \pi\hbar$

- All at a unit of time

Eigenvector Normalization Check

- The denominators of our normalized eigenvectors come from their squared magnitudes
- **For λ = -1:**
- $\|U_{-1}\| = 1 + \left(-\sqrt{2} - 1\right)^2 = 1 + (1 + 2 + 2\sqrt{2})$=4+2$\sqrt{2}$
- **For λ = 1:**
- $\|U_{-2}\| = 1 + \left(\sqrt{2} - 1\right)^2 = 1 + (1 + 2 - 2\sqrt{2})$=4-2$\sqrt{2}$

Understanding Initial State Components

- Projecting an Arbitrary State onto Eigenstates
- Any initial state $|\Psi >= \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ can be expressed as

$$|\Psi >= c_1|\psi_1 > +c_2|\psi_2 >$$

Projection onto $\boldsymbol{\lambda} = +\mathbf{1}$ eigenvector:

$$\frac{1}{\sqrt{4-2\sqrt{2}}}\begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix} \rightarrow \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{bmatrix} \alpha \\ (\sqrt{2}-1)\beta \end{bmatrix}$$

$$\boldsymbol{c_1} = \frac{\mathbf{1}}{\sqrt{\mathbf{4}-\mathbf{2}\sqrt{\mathbf{2}}}}[\boldsymbol{\alpha}+(\sqrt{\mathbf{2}}-\mathbf{1})\boldsymbol{\beta}]$$

Understanding Initial State Components

- **Projection onto $\boldsymbol{\lambda} = -\mathbf{1}$ eigenvector:**

$$\frac{1}{\sqrt{4-2\sqrt{2}}}\begin{bmatrix} 1 \\ -\sqrt{2}-1 \end{bmatrix} \rightarrow \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{bmatrix} \alpha \\ -(\sqrt{2}-1)\beta \end{bmatrix}$$

$$\boldsymbol{c_2} = \frac{\mathbf{1}}{\sqrt{\mathbf{4}+\mathbf{2}\sqrt{\mathbf{2}}}}[\boldsymbol{\alpha} - (\sqrt{\mathbf{2}} - \mathbf{1})\boldsymbol{\beta}]$$

Understanding Initial State Components

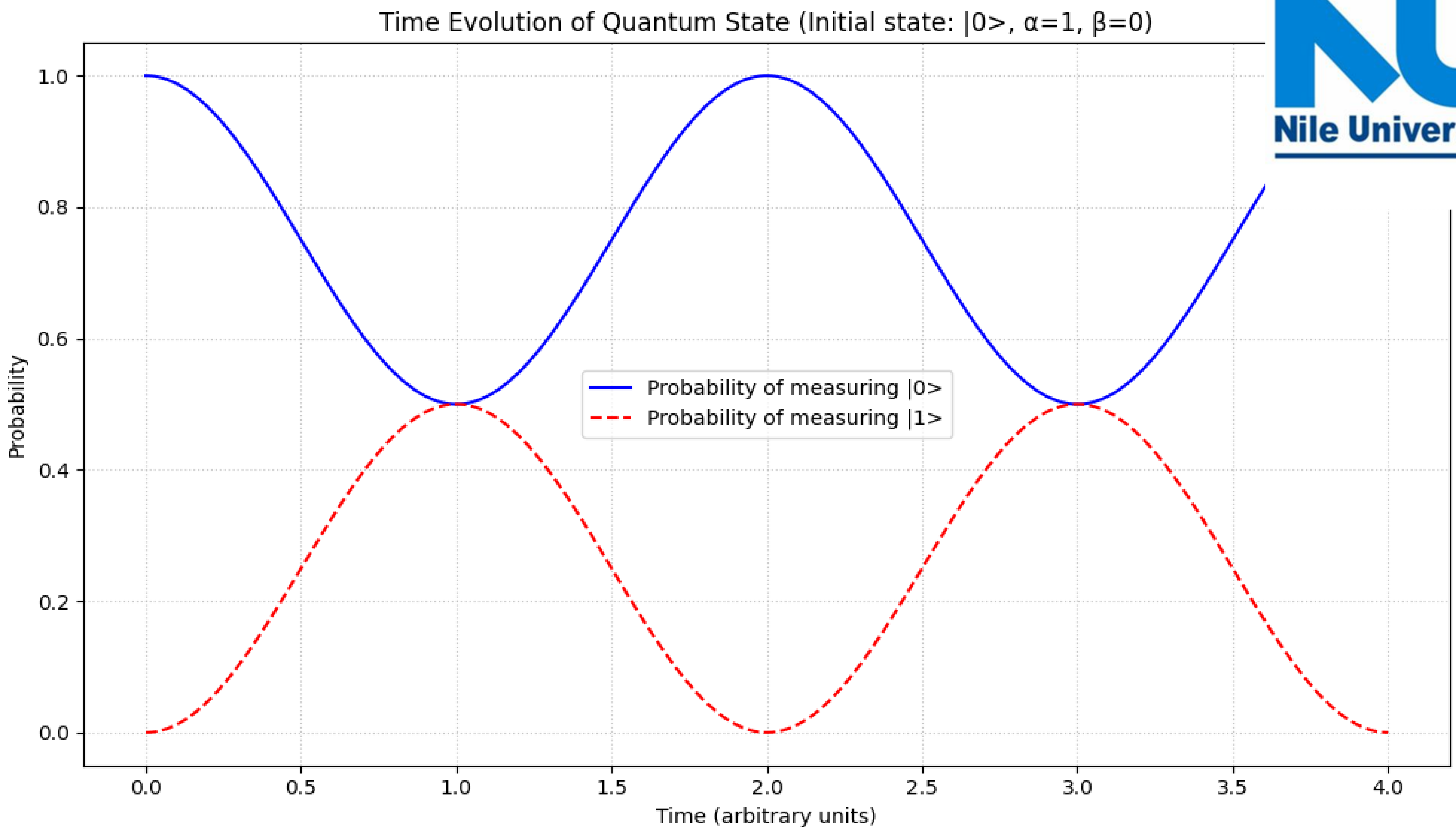
- At t=0 $$|\Psi(0)>= \begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

$$\frac{a+\beta(\sqrt{2}-1)}{\sqrt{4-2\sqrt{2}}}\begin{bmatrix}1\\ \sqrt{2}-1\end{bmatrix}+\frac{a+\beta(-\sqrt{2}-1)}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix}1\\ -\sqrt{2}-1\end{bmatrix}.e^{-i\pi t}$$

To simplify, bring both terms to a **common denominator of 8 by conjugate multiplication** :

$$|\Psi(0)>= \left[\frac{2(4+2\sqrt{2})\alpha+(4+2\sqrt{2})B(\sqrt{2}-1)}{8}\right]\begin{bmatrix}1\\ \sqrt{2}-1\end{bmatrix}+\left[\frac{2(4-2\sqrt{2})\alpha+(4-2\sqrt{2})B(-\sqrt{2}-1)}{8}\right]\begin{bmatrix}1\\ -(\sqrt{2}-1)\end{bmatrix}e^{-i\pi t}$$

$$|\Psi(0)>= \frac{1}{\sqrt{2}}\begin{bmatrix}a+\beta\\ a-\beta\end{bmatrix}$$



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Time Evolution of an Arbitrary State

- In quantum mechanics, a system's state evolves over time. If we know its energy eigenstates, we can represent any initial state as a mix of them and track its evolution

$$|\Psi(t) > = c_1|\psi_1 > e^{-iE_1 t/\hbar} + c_2|\psi_2 > e^{-iE_2 t/\hbar}$$

$\psi_1 >/\psi_2>$:energy eigenstates of the Hamiltonian.
E_1/ E_2 : their corresponding energy eigenvalues.
c_1/c_2: complex coefficients that define the initial state.
$\hbar$:reduced Planck constant.
t : time

It describe each component of the initial state acquires a **time-dependent phase factor**.

Time Evolution of an Arbitrary State

$$|\Psi(t)> = c_1|\psi_1> e^{-iE_1t/\hbar} + c_2|\psi_2> e^{-iE_2t/\hbar}$$

- The total wavefunction $|\Psi(t) >$ is a **superposition** of energy eigenstates.
- Each eigenstate evolves at a **different rate** depending on its energy E_n
- The result is a kind of **quantum rotation or interference** as time progresses.
- This leads to phenomena like **quantum beating, oscillations**, and **state transitions**.

Time Evolution of an Arbitrary State

- $E_1 = 0 \; E_2 = \pi\hbar$
- Substitute into the time evolution equation:

$$|\Psi(t) >= c_1 \left|\psi_1 > e^{-iE_1 t/\hbar} \quad + c_2 \right|\psi_2 > e^{-iE_2 t/\hbar}$$
$$|\Psi(t) >= c_1 |\psi_1 + c_2 |\psi_2 > e^{-i\pi t}$$

$$\sum_{n=1}^{2} C_n \left|\Psi n > e^{-iE_n t/\hbar}\right.$$